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T.H. KOORNWINDER & I.G. SPRINKHUIZEN-KUYPER

HYPERGEOMETRIC FUNCTIONS OF 2 × 2 MATRIX ARGUMENT ARE SOLUTIONS OF THE PARTIAL DIFFERENTIAL EQUATIONS FOR APPELL's FUNCTION F_4

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Hypergeometric functions of 2×2 matrix argument are solutions of the partial differential equations for Appell's function F_4 *)

by

T.H. Koornwinder & I.G. Sprinkhuizen-Kuyper.

ABSTRACT

The statement in the title is proved. As a result the hypergeometric function of 2×2 matrix argument is written as a linear combination of two hypergeometric Appell functions F_4.

KEY WORDS & PHRASES: *hypergeometric functions of matrix argument; Appell's hypergeometric function* F_4.

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*) This paper is not for review; it is meant for publication elsewhere.

1. INTRODUCTION

Hypergeometric functions of matrix argument, introduced by HERZ [6] are of considerable interest in multivariate statistics (cf. for instance CONSTANTINE [2] and JAMES [7]). Until now the hypergeometric functions of matrix argument on the one hand and of Appell and Lauricella type on the other hand seemed to be completely different generalizations of the one-variable case. In the present note it will be shown that at least in the two variable case some relationship exists between these generalized hypergeometric functions. It will be proved that both the hypergeometric function ${}_2F_1(a,b;c;Z)$ of 2×2 matrix argument Z and Appell's hypergeometric function in two variables $F_4(a,b;c-\frac{1}{2},1+a+b-c;x,y)$ are solutions of the same set of partial differential equations. Since a fundamental system of solutions around $(0,0)$ of the partial differential equations for F_4 is known, there follows an explicit expression of ${}_2F_1(a,b;c;Z)$ as a linear combination of two F_4 functions. For literature about F_4 functions see ERDÉLYI [4, Sect.5.7].

The idea that the results presented here would be valid, originated from our research on a class of orthogonal polynomials in two variables (cf. KOORNWINDER & SPRINKHUIZEN [9]). In a forthcoming paper we will extend the results presented here to the solutions of the partial differential equations of F_4 with arbitrary parameters, thus continuing the work initiated by ERDÉLYI [3].

2. PRELIMINARIES

CONSTANTINE [2] gives a series expansion for ${}_pF_q(Z)$ in terms of so-called zonal polynomials which were introduced by JAMES [7]. If Z is a 2×2 matrix then the zonal polynomials are explicitly known (cf. JAMES [8,(7.9)]. This results in the explicit expansion

$$(2.1) \qquad {}_2F_1\left(a,b;c;\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\right) =$$
$$= \sum_{m=0}^{\infty} \sum_{\ell=0}^{m} \frac{(a)_m (a-\frac{1}{2})_\ell (b)_m (b-\frac{1}{2})_\ell (\frac{3}{2})_{m-\ell}}{(c)_m (c-\frac{1}{2})_\ell (\frac{3}{2})_m \quad \ell! \quad (\frac{1}{2})_{m-\ell}} (XY)^{\frac{1}{2}(m+\ell)} P_{m-\ell}\left(\frac{1}{2}\frac{X+Y}{\sqrt{XY}}\right),$$

$a, b, c \in \mathbb{C}$, $c \neq \frac{1}{2}, 0, -\frac{1}{2}, -1, \ldots$, $X, Y \in \mathbb{C}$, $|X|, |Y| < 1$, where $P_n(x)$ denotes the Legendre polynomial of degree n (cf. ERDÉLYI [5]). Note that

$$ {}_2F_1\left(a,b;c;\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}\right) = \sum_{m=0}^{\infty} \frac{(a)_m (b)_m}{(c)_m \, m!} X^m = {}_2F_1(a,b;c;X). $$

For ${}_2F_1\left(a,b;c;\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\right)$ MUIRHEAD [10] gives the following set of partial differential equations

(2.2a) $$ X(1-X)z_{XX} + \{c-\tfrac{1}{2}-(a+b+\tfrac{1}{2})X+\tfrac{1}{2}\frac{X(1-X)}{X-Y}\}z_X - \tfrac{1}{2}\,\frac{Y(1-Y)}{X-Y}\,z_Y = abz, $$

(2.2b) $$ Y(1-Y)z_{YY} + \tfrac{1}{2}\,\frac{X(1-X)}{X-Y}\,z_X + \{c-\tfrac{1}{2}-(a+b+\tfrac{1}{2})Y-\tfrac{1}{2}\frac{Y(1-Y)}{X-Y}\}z_Y = abz, $$

and he proves that ${}_2F_1\left(a,b;c;\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\right)$ is the unique solution of each of these differential equations that satisfies the additional properties

(a) $z(X,Y)$ is a symmetrical function of X and Y, and

(b) z is regular in (0,0) and $z(0,0) = 1$.

The hypergeometric function $F_4(\alpha,\beta;\gamma,\gamma';x,y)$ of two variables is defined by

(2.3) $$ F_4(\alpha,\beta;\gamma,\gamma';x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_m(\gamma')_n m!\,n!} x^m y^n, $$

$\alpha, \beta, \gamma, \gamma' \in \mathbb{C}$, $\gamma, \gamma' \neq 0, -1, -2, \ldots$, $x, y \in \mathbb{C}$, $|x|^{\frac{1}{2}} + |y|^{\frac{1}{2}} < 1$, cf. APPELL & KAMPÉ DE FÉRIET [1] or SLATER [11]. Note that

$$ F_4(\alpha,\beta;\gamma,\gamma';0,y) = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n}{(\gamma')_n n!} y^n = {}_2F_1(\alpha,\beta;\gamma';y). $$

The function $F_4(\alpha,\beta;\gamma,\gamma'; x,y)$ satisfies the following set of partial differential equations

(2.4a) $$ x(1-x)z_{xx} - 2xyz_{xy} - y^2 z_{yy} + (\gamma-(\alpha+\beta+1)x)z_x - (\alpha+\beta+1)yz_y = \alpha\beta z, $$

$$(2.4b)\qquad -x^2z_{xx} - 2xyz_{xy} + y(1-y)z_{yy} - (\alpha+\beta+1)xz_x + (\gamma'-(\alpha+\beta+1)y)z_y = \alpha\beta z.$$

In any simply connected region included in $\{(x,y) \in \mathbb{C}^2 \mid x \neq 0,\ y \neq 0,\ |x|^{\frac{1}{2}} + |y|^{\frac{1}{2}} < 1\}$ the general solution of (2.4a,b) reads

$$(2.5)\qquad \begin{aligned} z(x,y) = {} & AF_4(\alpha,\beta;\gamma,\gamma';x,y) + \\ & + By^{1-\gamma'}F_4(1+\alpha-\gamma',1+\beta-\gamma';\gamma,2-\gamma';x,y) + \\ & + Cx^{1-\gamma}F_4(1+\alpha-\gamma,1+\beta-\gamma;2-\gamma,\gamma';x,y) + \\ & + Dx^{1-\gamma}y^{1-\gamma'}F_4(2+\alpha-\gamma-\gamma',2+\beta-\gamma-\gamma';2-\gamma,2-\gamma';x,y), \end{aligned}$$

provided that γ and γ' are noninteger (cf. APPELL & KAMPÉ DE FÉRIET [1]).

3. THE MAIN RESULT

Consider the transformation of variables

$$(3.1)\qquad \begin{cases} x = XY, \\ y = (1-X)(1-Y). \end{cases}$$

THEOREM 3.1. *The function given by* (2.1) *is an analytic function of* x *and* y *in some neighbourhood of* $(0,1)$ *and it satisfies the system of partial differential equations* (2.4a,b) *with* $\alpha := a$, $\beta := b$, $\gamma := c - \frac{1}{2}$, $\gamma' := 1 + a + b - c$.

PROOF. The first statement is clear from (2.1). For proving the second statement let S and T denote the equations (2.2a) and (2.2b), respectively. Then equations (2.4a) and (2.4b) are obtained by transforming the equations

$$\tfrac{1}{2}(S+T) - \frac{2-X-Y}{2(X-Y)}(S-T) \text{ and}$$

$$\tfrac{1}{2}(S+T) + \frac{X+Y}{2(X-Y)}(S-T)$$

in terms of x and y. □

THEOREM 3.2.

$$
{}_2F_1\left(a,b;c;\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\right) \doteq
$$

$$
= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} F_4(a,b;c-\tfrac{1}{2},1+a+b-c;XY,(1-X)(1-Y))
$$

$$
+ \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)} \{(1-X)(1-Y)\}^{c-a-b} F_4(c-b,c-a;c-\tfrac{1}{2},1+c-a-b;
$$

$$
XY,(1-X)(1-Y)),
$$

with X *and* Y *such that all terms are well-defined and provided that* c *is not an integer or half integer* $\leq \frac{1}{2}$ *and that* $a + b - c$ *is not an integer.*

PROOF. Let $z(x,y)$ denote the function given by (2.1) considered as a function of x and y in a neighbourhood of (0,1). Then, because of (2.5) and since z is regular in (0,1), it follows that

$$
\begin{aligned}
z = A\ & F_4(a,b;c-\tfrac{1}{2},1+a+b-c;x,y) + \\
& + B\ y^{c-a-b} F_4(c-b,c-a;c-\tfrac{1}{2},1+c-a-b;x,y).
\end{aligned}
$$

Restriciton to $x = 0$ gives

$$
\begin{aligned}
{}_2F_1(a,b;c;1-y) = A\ & {}_2F_1(a,b;1+a+b-c;y) + \\
& + B\ y^{c-a-b}\ {}_2F_1(c-b,c-a;1+c-a-b;y).
\end{aligned}
$$

It follows from ERDÉLYI [4,§2.10] that

$$
A = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad B = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}. \quad \square
$$

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